Combinatorial problems on zero-sum sequences

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Outline

1. Zero-sum sequences
2. Cyclic sieving phenomenon
3. Cayley table

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Zero-sum sequences

Classical zero-sum theory: *long sequence* contains zero-sum subsequences with prescribed properties. (a Ramsey-type problem)

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In this talk, we will not discuss classical zero-sum theory but present some connection between zero-sum sequences with

- combinatorial reciprocity
- cyclic sieving phenomenon
- Cayley table of abelian groups (Latin squares)

Zero-sum sequences

Let G be an additive finite abelian group. By a *sequence*

$$S = g_1 \cdot \ldots \cdot g_k$$

over G, we mean a finite sequence of elements $g_1, \ldots, g_k \in G$ which is **unordered** and **repetition** of terms is allowed, and k is called the *length* of S.

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$$\sigma(S) = g_1 + \cdots + g_k.$$

We say that S is a *zero-sum sequence* if $\sigma(S)$ equals 0_G, the identity element of G.

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For example, $S = \overline{1} \cdot \overline{1} \cdot \overline{2}$ is a zero-sum sequence over $\mathbb{Z}_4$ of length 3.

Number of zero-sum sequences

We define the set

$$\mathsf{M}(G,k) = \{S \text{ is a sequence over } G \mid \sigma(S) = 0_G,\ |S| = k\}.$$

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For example, we consider $\mathbb{Z}_4$. Then we have

$$\mathsf{M}(\mathbb{Z}_4, 3) = \{S_1, S_2, S_3, S_4, S_5\},$$

where

$$S_1 = \overline{0} \cdot \overline{0} \cdot \overline{0}$$

$$S_2 = \overline{0} \cdot \overline{1} \cdot \overline{3}$$

$$S_3 = \overline{0} \cdot \overline{2} \cdot \overline{2}$$

$$S_4 = \overline{1} \cdot \overline{1} \cdot \overline{2}$$

$$S_5 = \overline{2} \cdot \overline{3} \cdot \overline{3}.$$

Next, we consider $\mathbb{Z}_3$. Then we have

$$\mathsf{M}(\mathbb{Z}_3, 4) = \{T_1, T_2, T_3, T_4, T_5\},$$

where

$$\begin{aligned}
T_1 &= \overline{0} \cdot \overline{0} \cdot \overline{0} \cdot \overline{0} \\
T_2 &= \overline{0} \cdot \overline{0} \cdot \overline{1} \cdot \overline{2} \\
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\end{aligned}$$

A reciprocity on cyclic groups

Note that

$$|\mathsf{M}(\mathbb{Z}_4, 3)| = |\mathsf{M}(\mathbb{Z}_3, 4)|.$$

[1]M. Fredman, *A symmetry relationship for a class of partitions*, J. Combin. Theory Ser. A, 18 (1975), pp. 199–202.

[2]A.G. Elashvili, M. Jibladze and D. Pataraia, *Combinatorics of Necklaces and Hermite Reciprocity*, J. Algebraic Comb., 10 (1999), pp. 173–188.

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Using methods from **generating functions**, **necklace interpretation**, Freedman (1975)[1], N. Alon, G. Andrews, R. Stanley (1990s, mentioned by Elashvili, Jibladze and Pataraia [2]) independently proved that

$$|\mathsf{M}(\mathbb{Z}_n, m)| = |\mathsf{M}(\mathbb{Z}_m, n)|$$

holds for any positive integers n, m.

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Theorem 1.1 (Han and Zhang, SIAM J. Discrete Math. 2021)

For any two abelian groups G and H with $(|G|, |H|) = 1$, we have

$$|\mathsf{M}(G, |H|)| = |\mathsf{M}(H, |G|)|.$$

Zero-sum sequences over finite abelian groups

Theorem 1.1 (Han and Zhang, SIAM J. Discrete Math. 2021)

For any two abelian groups G and H with $(|G|,|H|)=1$, we have

$$|\mathsf{M}(G,|H|)| = |\mathsf{M}(H,|G|)|.$$

Moreover, the above cardinality is

$$\mathsf{Cat}_{|G|,|H|} = \frac{1}{|G|+|H|}\binom{|G|+|H|}{|G|,|H|},$$

the rational Catalan numbers.

We provide a **combinatorial interpretation** of this result.

Rational Catalan number

Let a, b be positive integers with $(a, b) = 1$, we call

$$\mathsf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a,b}$$

the rational Catalan number.

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Rational Catalan number

Let a, b be positive integers with $(a, b) = 1$, we call

$$\mathsf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a,b}$$

the rational Catalan number. In particular,

$$\mathsf{Cat}_{n,n+1} = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$$

is the classical Catalan number, which counts at least 200 different objects.[3]

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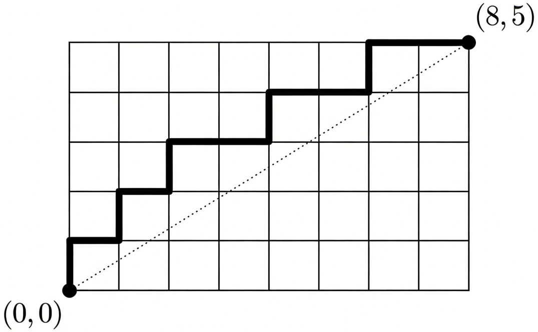
Rational Dyck paths

In general, for any a, b with $(a, b) = 1$, we consider lattice paths from $(0, 0)$ to (a, b) which only use unit steps $(1, 0)$ or $(0, 1)$ and stay above the diagonal line $y = \frac{b}{a}x$. It turns out that the number of such paths are $\mathsf{Cat}_{a,b} = \frac{1}{a+b}\binom{a+b}{a,b}$.



A reciprocity

Theorem 1.2 (Li and Z., Electronic J. Combin. 2024)

Let G and H be finite abelian groups. We have

$$|\mathsf{M}(G, |H|)| = |\mathsf{M}(H, |G|)|$$

if and only if

$$\varphi_G(d) = \varphi_H(d), \ \textit{for any } d \mid (|G|, |H|).$$

Here, $\varphi_G(d)$ is the number of elements of order d in G.

Combinatorial reciprocity

The famous **combinatorial reciprocity** (R.P. Stanley, Adv. Math. 1974) is a fascinating phenomenon in combinatorics where a function $f(n)$ that counts combinatorial objects for positive integer inputs can be reinterpreted, when evaluated at negative integers $f(-n)$, seems meaningless but counts another class of objects (often with a sign).

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Let $f(n) = \binom{n}{k}$, then

$$f(-n) = \binom{-n}{k} = (-1)^k \binom{n+k-1}{k},$$

where for $\alpha \in \mathbb{R}$,

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

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Note that, $\binom{n}{k}$ counts the k-**subset** and $\binom{n+k-1}{k}$ counts the k-**multisubset**.

Combinatorial reciprocity

Combinatorial reciprocity appeared in many classical combinatorial objects

- Ehrhart polynomial (in discrete geometry)
- chromatic polynomial (in graph theory)
- order polynomials (in partial ordered set)
- and so on...

and reveals deep and unexpected structure.[4]

[4] M. Beck and R. Sanyal, *Combinatorial reciprocity theorems*, Grad. Stud. Math., 195, American Mathematical Society, Providence, RI, 2018, xiv+308 pp.

Combinatorial reciprocity - Ehrhart-Macdonald reciprocity

For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, the convex hull of finitely many points with integer coordinates,

[5] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-point Enumeration in Polyhedra*, Undergrad. Texts Math., Springer, New York, 2015, xx+285 pp.

Combinatorial reciprocity - Ehrhart-Macdonald reciprocity

For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, the convex hull of finitely many points with integer coordinates, Ehrhart proved that the counting function

$$L_{\mathcal{P}}(t) = |t\mathcal{P} \cap \mathbb{Z}^d|, \qquad t \in \mathbb{N},$$

is a polynomial in t of degree d, where $t\mathcal{P} = \{tx \mid x \in \mathcal{P}\}$. The polynomial $L_{\mathcal{P}}(t)$ is called the *Ehrhart polynomial.*[5]

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Ehrhart-Macdonald reciprocity states that:

$$L_{\mathcal{P}}(-t) = (-1)^d L_{\mathcal{P}^\circ}(t), \qquad t \in \mathbb{N},$$

where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

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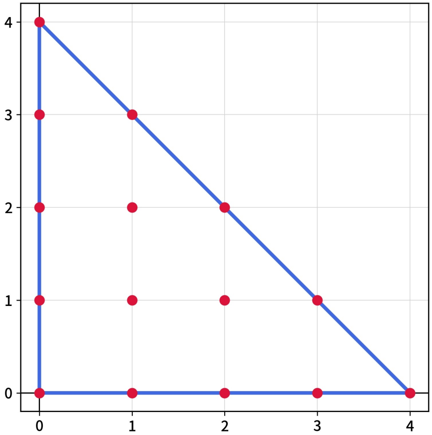
$\mathcal{P} = \text{conv}\{(0,0),(1,0),(0,1)\}$ (the triangle) and $t = 4$

Note that, $L_{\mathcal{P}}(t) = \frac{(t+1)(t+2)}{2}$, $L_{\mathcal{P}}(4) = 15$, and $L^{\circ}_{\mathcal{P}}(4) = 3 = (-1)^2 L_{\mathcal{P}}(-4)$.



Combinatorial reciprocity for $\mathsf{M}(G, m)$

What happens if we consider $|\mathsf{M}(G, -m)|$?

[6] D. C. Han and H. B. Zhang, *A reciprocity on finite abelian groups involving zero-sum sequences*, SIAM J. Discrete Math., 35(2) (2021), 1077–1095.

Combinatorial reciprocity for $\mathsf{M}(G, m)$

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Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ be a finite abelian group with $|G| = n = n_1 \cdots n_r$ and $n_1 | \cdots | n_r$. Then for any positive integer m, we have

$$|\mathsf{M}(G, m)| = \frac{1}{n} \sum_{d|(n,m)} \varphi_G(d) \binom{n/d + m/d - 1}{n/d - 1},$$

where $\varphi_G(d) = \sum_{l|d} \mu(d/l) \prod_{i=1}^{r} (n_i, l)$ is the number of elements of order d in G. (for a proof using invariant theory, see[6])

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$|\mathsf{M}(G, -m)|$ can be calculated using $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$, where $\alpha \in \mathbb{R}$.

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Explicit Counting formula

Let

$$\mathsf{M}^\circ(G,m) = \{S \mid S \in \mathsf{M}(G,m) \text{ and every elements of } G \text{ appear in } S\}.$$

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For example, we have

$$\mathsf{M}^\circ(\mathbb{Z}_3, 5) = \{T_1, T_2\},$$

where

$$T_1 = \overline{0}\cdot\overline{0}\cdot\overline{0}\cdot\overline{1}\cdot\overline{2}$$

$$T_2 = \overline{0}\cdot\overline{1}\cdot\overline{1}\cdot\overline{2}\cdot\overline{2}.$$

Combinatorial reciprocity

Theorem 1.3 (Han-Wang-Zhang-Zhang, 2026)

Let G be an abelian group of order n, and $m \geq 1$, then we have

$$|\mathsf{M}(G, -m)| = (-1)^{n-1}|\mathsf{M}^{\circ}(G, m)|. \tag{1}$$

Therefore, $\mathsf{M}(G, m)$ also exhibits an interesting combinatorial reciprocity.

Combinatorial reciprocity

Theorem 1.3 (Han-Wang-Zhang-Zhang, 2026)

Let G be an abelian group of order n, and $m \geq 1$, then we have

$$|\mathsf{M}(G,-m)| = (-1)^{n-1}|\mathsf{M}^{\circ}(G,m)|. \tag{1}$$

Therefore, $\mathsf{M}(G,m)$ also exhibits an interesting combinatorial reciprocity.

For example,

$$|\mathsf{M}(\mathbb{Z}_3,m)| = \begin{cases} \frac{1}{3}\binom{m+2}{2}, & \text{if } 3 \nmid m, \\ \frac{1}{3}\left(\binom{m+2}{2}+2\right), & \text{if } 3 \mid m. \end{cases}$$

Therefore, we have

$$|\mathsf{M}(\mathbb{Z}_3,-5)| = \frac{1}{3}\binom{-5+2}{2} = \frac{1}{3}\frac{(-3)(-4)}{2} = 2 = (-1)^{3-1}|\mathsf{M}^{\circ}(\mathbb{Z}_3,5)|.$$

Combinatorial reciprocity

For $\tau \in \mathsf{Aut}(G)$, note that for a zero-sum sequence $S = g_1 \cdots g_k$, $\tau(S) = \tau(g_1) \cdots \tau(g_k)$ is also zero-sum. Let

$$\mathsf{M}(G,m)^\tau = \{S \mid S \in \mathsf{M}(G,m),\ \tau(S) = S\}$$

and

$$\mathsf{M}^\circ(G,m)^\tau = \{S \mid S \in \mathsf{M}(G,m),\ \tau(S) = S,\ \mathsf{supp}(S) = G\}.$$

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Theorem 1.4 (Han-Wang-Zhang-Zhang, 2026)

Let G be an abelian group of order n, $\tau \in \mathsf{Aut}(G)$, $\mathcal{O}(\tau)$ the number of orbits of τ and $m \geq 1$, then we have

$$|\mathsf{M}(G, -m)^\tau| = (-1)^{\mathcal{O}(\tau)-1} |\mathsf{M}^\circ(G, m)^\tau|.$$

Outline

Cyclic sieving phenomenon

Let X be a finite set, carries an action of a cyclic group $C_n = \langle c \rangle$ of order n, Let $f(q)$ be a polynomial in q having non-negative integer coefficients, with the property that $f(1) = |X|$. Let ζ_n be a primitive n-th root of unity.

A triple

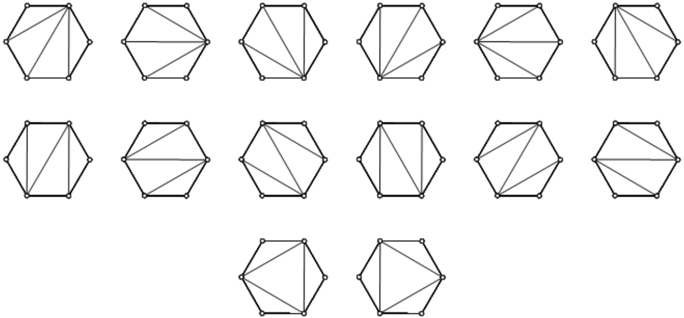
$$(X, C_n, f(q))$$

exhibits the **cyclic sieving phenomenon** (Reiner-Stanton-White, JCTA 2004) if for any positive integer d,

$$|\{x \mid c^d \cdot x = x,\ x \in X\}| = f(\zeta_n^d).$$

Cyclic sieving phenomenon

Let X_{n+2} be the set of triangulations of a regular $(n+2)$-gon. Then it is well known that $|X_{n+2}| = \mathsf{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$. For example, X_6 contains the following:



Cyclic sieving phenomenon

The cyclic group $C_{n+2} = \langle c \rangle$ acting on X_{n+2} by rotation.

Let

$$\mathsf{Cat}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

be the MacMahon's q-Catalan polynomial, where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]!_q = [n]_q [n-1]_q \cdots [1]_q$.

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be the MacMahon's q-Catalan polynomial, where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]!_q = [n]_q[n-1]_q \cdots [1]_q$. For example,

$$\mathsf{Cat}_4(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}.$$

$$|\{x \mid c^2 \cdot x = x, x \in X_6\}| = 2 = \mathsf{Cat}_4(\zeta_6^2)$$
$$|\{x \mid c^3 \cdot x = x, x \in X_6\}| = 6 = \mathsf{Cat}_4(\zeta_6^3)$$

Then the triple $(X_{n+2}, C_{n+2}, \mathsf{Cat}_n(q))$ exhibits the CSP (Reiner-Stanton-White, JCTA 2004).

Rational q-Catalan number

Recall that for $|G| = n$ and $(n, m) = 1$, we have $|\mathsf{M}(G, m)| = \mathsf{Cat}_{n,m}$, where

$$\mathsf{Cat}_{n,m} = \frac{1}{n+m}\binom{n+m}{n, m}$$

the rational Catalan number.

[7] M. Haiman, *Hilbert schemes, polygraphs and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14 (2001), no. 4, 941–1006.

[8] M. Haiman, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Invent. Math. 149 (2002), 371–407.

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the rational Catalan number. For positive integers n, m with $(n, m) = 1$, let

$$\mathsf{Cat}_{n,m}(q) = \frac{1}{[n+m]_q}\begin{bmatrix} n+m \\ m \end{bmatrix}_q.$$

It is known that $\mathsf{Cat}_{n,m}(q)$ a polynomial in q with non-negative integer coefficients; see [7], [8]. Therefore, $\mathsf{Cat}_{n,m}(q)$ provides a q-analog of $|\mathsf{M}(G, m)|$ (where $|G| = n$ and $(n, m) = 1$).

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Cyclic sieving phenomenon

Theorem 2.1 (Han-Wang-Zhang-Zhang, 2026)

Let G be an abelian group of odd order n. Let $X = \mathsf{M}(G, m)$ (where $(n, m) = 1$) and $C_2 = \langle c \rangle$ acting on X with

$$c \cdot (g_1 \cdot g_2 \cdots g_m) := (-g_1) \cdot (-g_2) \cdots (-g_m),$$

and

$$\mathsf{Cat}_{n,m}(q) = \frac{1}{[n+m]_q} \begin{bmatrix} n+m \\ m \end{bmatrix}_q .$$

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Then $(\mathsf{M}(G, m), C_2, \mathsf{Cat}_{n,m}(q))$ exhibits the CSP.

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$$\mathsf{Cat}_{n,m}(q) = \frac{1}{[n+m]_q} \begin{bmatrix} n+m \\ m \end{bmatrix}_q .$$

Then $(\mathsf{M}(G, m), C_2, \mathsf{Cat}_{n,m}(q))$ exhibits the CSP.

The C_2-action case of CSP is also called the $q = -1$ phenomenon, first introduced by Stembridge (Duke Math. J. 1994).

Cyclic sieving phenomenon

$$\mathsf{M}(\mathbb{Z}_5,3)=\{\overline{0}\cdot\overline{0}\cdot\overline{0},\ \overline{1}\cdot\overline{2}\cdot\overline{2},\ \overline{0}\cdot\overline{1}\cdot\overline{4},\ \overline{0}\cdot\overline{2}\cdot\overline{3},\ \overline{1}\cdot\overline{1}\cdot\overline{3},\ \overline{2}\cdot\overline{4}\cdot\overline{4},\ \overline{3}\cdot\overline{3}\cdot\overline{4}\}$$

$$\mathsf{Cat}_{5,3}(q)=\frac{1}{[8]_q}\frac{[8]!_q}{[5]!_q[3]!_q}=1+q^2+q^3+q^4+q^5+q^6+q^8$$

satisfies $\mathsf{Cat}_{5,3}(1)=|\mathsf{M}(\mathbb{Z}_5,3)|$.

$$\begin{aligned}\overline{1}\cdot\overline{2}\cdot\overline{2}&\longrightarrow\overline{3}\cdot\overline{3}\cdot\overline{4},\\ \overline{1}\cdot\overline{1}\cdot\overline{3}&\longrightarrow\overline{2}\cdot\overline{4}\cdot\overline{4},\\ \overline{2}\cdot\overline{4}\cdot\overline{4}&\longrightarrow\overline{1}\cdot\overline{1}\cdot\overline{3},\\ \overline{3}\cdot\overline{3}\cdot\overline{4}&\longrightarrow\overline{1}\cdot\overline{2}\cdot\overline{2}\end{aligned}\qquad\begin{aligned}\overline{0}\cdot\overline{0}\cdot\overline{0}&\longrightarrow\overline{0}\cdot\overline{0}\cdot\overline{0}\\ \overline{0}\cdot\overline{1}\cdot\overline{4}&\longrightarrow\overline{0}\cdot\overline{1}\cdot\overline{4}\\ \overline{0}\cdot\overline{2}\cdot\overline{3}&\longrightarrow\overline{0}\cdot\overline{2}\cdot\overline{3}\end{aligned}$$

Note that

$$3=|\{S\in\mathsf{M}(\mathbb{Z}_5,3)\mid c\cdot S=S\}|=\mathsf{Cat}_{5,3}(-1)=3$$

which exhibits CSP.

Cyclic sieving phenomenon

Theorem 2.2 (Han-Wang-Zhang-Zhang, 2026)

Let G be a finite abelian group of order n, let $m \in \mathbb{N}$ with $(n,m) = 1$, and let $c \in \mathrm{Aut}(G)$ have order d. Assume that c is semiregular on $G \setminus \{0\}$ with

$$c \cdot (g_1 \cdot g_2 \cdots g_m) := c(g_1) \cdot c(g_2) \cdots c(g_m).$$

Then the triple

$$\big(\mathsf{M}(G,m),\ \langle c\rangle,\ \mathrm{Cat}_{n,m}(q)\big)$$

exhibits the cyclic sieving phenomenon.

Cyclic sieving phenomenon

Theorem 2.2 (Han-Wang-Zhang-Zhang, 2026)

Let G be a finite abelian group of order n, let $m \in \mathbb{N}$ with $(n,m)=1$, and let $c \in \mathrm{Aut}(G)$ have order d. Assume that c is semiregular on $G \setminus \{0\}$ with

$$c \cdot (g_1 \cdot g_2 \cdots g_m) := c(g_1) \cdot c(g_2) \cdots c(g_m).$$

Then the triple

$$\big(\mathsf{M}(G,m),\ \langle c \rangle,\ \mathrm{Cat}_{n,m}(q)\big)$$

exhibits the cyclic sieving phenomenon.

Let $c \in \mathrm{Aut}(G)$ have finite order d. We say that c is *semiregular on* $G \setminus \{0\}$ if

$$\mathrm{Fix}_G(c^j) = \{0\} \qquad \text{for every } 1 \leq j < d.$$

Equivalently, every nonzero orbit of the cyclic group $\langle c \rangle$ has size d.

Cyclic sieving phenomenon

Corollary 2.3 (Han-Wang-Zhang-Zhang, 2026)

Let p be an odd prime and $X = \mathsf{M}(\mathbb{Z}_p, m)$ (where $(p, m) = 1$) and $\mathrm{Aut}(\mathbb{Z}_p) = C_{p-1} = \langle \sigma \rangle$ acting on X with

$$\sigma \cdot (g_1 \cdot g_2 \cdots g_m) := \sigma(g_1) \cdot \sigma(g_2) \cdots \sigma(g_m),$$

Then $(\mathsf{M}(\mathbb{Z}_p, m), \mathrm{Aut}(\mathbb{Z}_p), \mathrm{Cat}_{p,m}(q))$ exhibits the CSP.

Cyclic sieving phenomenon

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Then $(\mathsf{M}(\mathbb{Z}_p, m), \mathrm{Aut}(\mathbb{Z}_p), \mathrm{Cat}_{p,m}(q))$ exhibits the CSP.

Corollary 2.4 (Han-Wang-Zhang-Zhang, 2026)

Let $G = (\mathbb{Z}/p\mathbb{Z})^r$ with p prime, let $\lambda \in (\mathbb{Z}/p\mathbb{Z})^\times$ have multiplicative order d, and let c be the automorphism $x \mapsto \lambda x$ of G. If $(p, m) = 1$, then

$$\left(\mathsf{M}(\mathbb{Z}_p^r, m),\ \langle c \rangle,\ \mathrm{Cat}_{p^r,m}(q)\right)$$

exhibits the cyclic sieving phenomenon.

Cyclic sieving phenomenon

There are also several studies on the automorphism of power monoids. To study the cyclic sieving phenomenon:

1. For a finite set X, find an interesting q-analog of X (like the q-Catalan numbers)
2. Consider a natural cyclic action on X, then check the cyclic sieving phenomenon.
3. One important reason to study cyclic sieving phenomenon is a representation theoretic explanation: find a natural vector space V with a basis indexed by X, where V is normally a representation space of the cyclic groups.
4. In many cases, the essence of the cyclic sieving phenomenon is an isomorphism between two representations of a cyclic group (one is permutation, the other one is diagonal).

Outline

Determinant of the circulant matrix

The determinant of the circulant matrix is a classical exercise in linear algebra:

$$\mathsf{circ}(x_0, x_1, \ldots, x_{n-1}) = \begin{pmatrix} x_0 & x_1 & x_2 & \cdots & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_0 \\ x_2 & x_3 & x_4 & \cdots & x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1} & x_0 & x_1 & \cdots & x_{n-2} \end{pmatrix}.$$

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$$\mathsf{det}(\mathsf{circ}(x_0, x_1, \ldots, x_{n-1})) = \prod_{k=0}^{n-1} (\sum_{j=0}^{n-1} \zeta_n^{jk} x_j), \qquad \zeta_n = e^{2\pi i/n}.$$

Determinant of the Cayley table

Let G be an additive abelian group of order n.

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For example: $C_3 = \mathbb{Z}/3\mathbb{Z} = \{x_0 = \overline{0},\ x_1 = \overline{1},\ x_2 = \overline{2}\}$

$$\mathcal{M}_{C_3} = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix}.$$

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$C_4 = \mathbb{Z}/4\mathbb{Z} = \{x_0 = \overline{0},\ x_1 = \overline{1},\ x_2 = \overline{2},\ x_3 = \overline{3}\}$

$$\mathcal{M}_{C_4} = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_0 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_0 & x_1 & x_2 \end{pmatrix}.$$

Determinant of the Cayley table

The determinant of $\mathcal{M}_G$ (called the **group determinant**), denoted by $\mathsf{det}(\mathcal{M}_G)$, is a homogeneous polynomial of degree n in x_i's:

$$\mathsf{det}(\mathcal{M}_{C_2}) = \mathsf{det}\begin{pmatrix} x_0 & x_1 \\ x_1 & x_0 \end{pmatrix} = x_0^2 - x_1^2.$$

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$$\mathsf{det}(\mathcal{M}_{C_3}) = \mathsf{det}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix} = -x_0^3 - x_1^3 - x_2^3 + 3x_0x_1x_2.$$

Dedekind and Frobenius

Dedekind wrote to Frobenius (March 25, 1896) a letter containing the definition of the group determinant and the factorization in the abelian case.

Dedekind also hinted at some computations in the non-abelian case that he had done (without including them). Upon Frobenius' insistence, he hesitatingly formulated a conjecture in a letter dated April 3, 1896, for an arbitrary finite group G.

Frobenius' Theorem, December 3, 1896

Let G be a finite group of order n. Then we have

$$\mathsf{det}(\mathcal{M}_G) = \prod_{i=1}^{r} P_i(x_g, \cdots)^{d_i},$$

where

1. r is the number of conjugacy classes of G;
2. P_i irreducible and $\deg(P_i) = d_i$. In particular, $\sum_{i=1}^{r} d_i^2 = |G|$.

When G is abelian and $\widehat{G} = \{\chi_0, \cdots, \chi_{n-1}\}$.

$$\mathsf{det}(\mathcal{M}_G) = \prod_{i=0}^{n-1} \Big(\sum_{g \in G}^{n-1} \chi_i(g) x_g \Big).$$

Permanent of the Cayley table

The permanent of $\mathcal{M}_G$ (called the **group permanent** of G), denoted by $\mathsf{per}(\mathcal{M}_G)$, is a homogeneous polynomial of degree n in x_i's:

$$\mathsf{per}(\mathcal{M}_{C_3}) = \mathsf{per}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix} = x_0^3 + x_1^3 + x_2^3 + 3x_0x_1x_2.$$

Recall that, for an $n \times n$ matrix $\mathcal{M} = (m_{ij})_{1 \le i,j \le n}$, the permanent

$$\mathsf{per}(\mathcal{M}) = \sum_{\tau \in S_n} \prod_{i=1}^{n} m_{i,\tau(i)}.$$

Group permanent

$$\mathrm{per}(\mathcal{M}_{C_4}) = \mathrm{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_0 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_0 & x_1 & x_2 \end{pmatrix}$$

$$= x_0^4 + x_1^4 + x_2^4 + x_3^4 + 4x_0^2x_1x_3 + 2x_0^2x_2^2$$

$$+ 4x_0x_1^2x_2 + 4x_0x_2x_3^2 + 4x_1x_2^2x_3 + 2x_1^2x_3^2.$$

Permanent of the Cayley table

Let $\mathcal{P}(G)$ denote the number of formally different monomials occurring in $\mathsf{per}(\mathcal{M}_G)$.

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Therefore, we have $\mathcal{P}(C_3) = 4$.

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$$\begin{aligned}\mathsf{per}(\mathcal{M}_{C_4}) = \mathsf{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_0 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_0 & x_1 & x_2 \end{pmatrix} \\ = x_0^4 + x_1^4 + x_2^4 + x_3^4 + 4x_0^2x_1x_3 + 2x_0^2x_2^2 \\ + 4x_0x_1^2x_2 + 4x_0x_2x_3^2 + 4x_1x_2^2x_3 + 2x_1^2x_3^2.\end{aligned}$$

Therefore, we have $\mathcal{P}(C_4) = 10$.

Permanent of the Cayley table

$$\mathsf{per}(\mathcal{M}_{C_5}) = \mathsf{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 & x_0 \\ x_2 & x_3 & x_4 & x_0 & x_1 \\ x_3 & x_4 & x_0 & x_1 & x_2 \\ x_4 & x_0 & x_1 & x_2 & x_3 \end{pmatrix}$$

$$\begin{aligned} &= x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5x_3x_4^3x_0 + 5x_3^2x_4x_0^2 + 5x_3^2x_4^2x_1 \\ &\quad + 5x_3^3x_0x_1 + 5x_4x_0^3x_1 + 5x_4^2x_0x_1^2 + 5x_4x_0^2x_1^2 + 5x_3x_4x_1^3 + 5x_3^3x_4x_2 \\ &\quad + 5x_4^2x_0^2x_2 + 5x_3x_0^3x_2 + 5x_4^3x_1x_2 + 5x_3^2x_1^2x_2 + 5x_4x_1^2x_2^2 + 5x_0x_1^3x_2 \\ &\quad + 5x_3x_4^2x_2^2 + 5x_3^2x_0x_2^2 + 5x_0^2x_1x_2^2 + 5x_4x_0x_2^3 + 5x_3x_1x_2^3 \\ &\quad + 15x_0x_1x_2x_3x_4. \end{aligned}$$

Therefore, we have $\mathcal{P}(C_5) = 26$.

Permanent of the Cayley table

$$\operatorname{per}(\mathcal{M}_{C_7}) = \operatorname{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_0 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_0 & x_1 \\ x_3 & x_4 & x_5 & x_6 & x_0 & x_1 & x_2 \\ x_4 & x_5 & x_6 & x_0 & x_1 & x_2 & x_3 \\ x_5 & x_6 & x_0 & x_1 & x_2 & x_3 & x_4 \\ x_6 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$

$$= x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7 + 7x_0^5x_1x_6 + 7x_0^4x_2x_6^2 + \cdots$$
$$+ 14x_0^4x_1x_2x_4 + 14x_0^3x_2^2x_5^2 + \cdots + 21x_0^3x_1x_3x_5^2 + 21x_0^2x_1^3x_5x_6 + \cdots$$
$$+ 35x_0^3x_2x_3x_4x_5 + \cdots + 42x_0^2x_3^2x_4x_5x_6 + \cdots + 49x_0^2x_2x_4^2x_5x_6 + \cdots$$
$$+ 133x_0x_1x_2x_3x_4x_5x_6.$$

We have $\mathcal{P}(C_7) = 246$.

Permanent and zero-sum sequences

Theorem 3.1 (M. Hall, Proc. Amer. Math. Soc., 1952)

For any finite abelian group G of order n, a monomial

$$\prod_{g\in G} x_g^{i_g}$$

appeared in $\mathsf{per}(\mathcal{M}_G)$ *if and only if* $\prod_{g\in G} g^{i_g}$ *is a* **zero-sum sequence** *of length n, i.e.,*

$$\sum_{g\in G} i_g g = 0_G,$$

the identity of G.

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the identity of G.

In our previous notation, $\mathcal{P}(G) = \mathsf{M}(G, |G|)$.

The group permanent determines the finite abelian group

Theorem 3.2 (Li and Z., Electronic J. Combin. 2024)

Let G and H be finite abelian groups. We have

$$\mathcal{P}(G) = \mathcal{P}(H) \;\Leftrightarrow\; G \cong H.$$

Determinant of the Cayley table

Let $\mathcal{D}et(G)$ be the number of different monomials in $\mathsf{det}(\mathcal{M}_G)$.

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Recall the group determinant:

$$\mathsf{det}(\mathcal{M}_{C_3}) = \mathsf{det}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix} = -x_0^3 - x_1^3 - x_2^3 + 3x_0x_1x_2.$$

We have, $\mathcal{D}et(C_3) = 4$.

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Recall the group determinant:

$$\mathsf{det}(\mathcal{M}_{C_3}) = \mathsf{det}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_0 \\ x_2 & x_0 & x_1 \end{pmatrix} = -x_0^3 - x_1^3 - x_2^3 + 3x_0x_1x_2.$$

We have, $\mathcal{D}et(C_3) = 4$.

In view of possible cancellations, we have $\mathcal{D}et(G) \leq \mathcal{P}(G)$.

Example - C_6, $\mathcal{P}(C_6) = 80$

$$\begin{aligned}
\mathrm{per}(Circ(x,y,z,t,u,v)) = {} & x^6 + y^6 + z^6 + t^6 + u^6 + v^6 + 6t^4uz + 6t^4vy \\
& + 3t^4x^2 + 6t^3u^2y + 12t^3uvx + 2t^3v^3 + 6t^3vz^2 \\
& + 12t^3xyz + 2t^3y^3 + 6t^2u^3x + 9t^2u^2v^2 + 9t^2u^2z^2 \\
& + 24t^2uvyz + 12t^2ux^2z + 18t^2uxy^2 + 18t^2v^2xz \\
& + 9t^2v^2y^2 + 12t^2vx^2y + 3t^2x^4 + 6t^2xz^3 + 9t^2y^2z^2 \\
& + 6tu^4v + 12tu^3yz + 24tu^2vxz + 12tu^2vy^2 \\
& + 18tu^2x^2y + 12tuv^3z + 24tuv^2xy + 12tuvx^3 \\
& + 12tuvz^3 + 24tuxyz^2 + 12tuy^3z + 6tv^4y + 6tv^3x^2 \\
& + 12tv^2yz^2 + 18tvx^2z^2 + 24tvxy^2z + 6tvy^4 \\
& + 6tx^2y^3 + 12tx^3yz + 6tyz^4 + 6u^4xz + 3u^4y^2 \\
& + 6u^3v^2z + 12u^3vxy + 2u^3x^3 + 2u^3z^3 + 6u^2v^3y \\
& + 9u^2v^2x^2 + 18u^2vyz^2 + 9u^2x^2z^2 + 12u^2xy^2z \\
& + 3u^2y^4 + 6uv^4x + 12uv^2xz^2 + 18uv^2y^2z \\
& + 24uvx^2yz + 12uvxy^3 + 6ux^4z + 6ux^3y^2 + 6uxz^4 \\
& + 6uy^2z^3 + 3v^4z^2 + 12v^3xyz + 2v^3y^3 + 6v^2x^3z \\
& + 9v^2x^2y^2 + 3v^2z^4 + 6vx^4y + 12vxyz^3 + 6vy^3z^2 \\
& + 2x^3z^3 + 9x^2y^2z^2 + 6xy^4z.
\end{aligned}$$

Example - C_6, $\mathcal{D}et(C_6) = 68$

$$
\begin{aligned}
\det(Circ(x,y,z,t,u,v)) \quad = \quad & x^6 - y^6 + z^6 - t^6 + u^6 - v^6 + 6t^4uz + 6t^4vy \\
& +3t^4x^2 - 6t^3u^2y - 12t^3uvx - 2t^3v^3 - 6t^3vz^2 \\
& -12t^3xyz - 2t^3y^3 + 6t^2u^3x + 9t^2u^2v^2 - 9t^2u^2z^2 \\
& +18t^2uxy^2 + 18t^2v^2xz - 9t^2v^2y^2 - 3t^2x^4 \\
& +6t^2xz^3 + 9t^2y^2z^2 - 6tu^4v + 12tu^3yz \\
& -18tu^2x^2y - 12tuv^3z + 12tuvx^3 + 12tuvz^3 \\
& -12tuy^3z + 6tv^4y - 6tv^3x^2 - 18tvx^2z^2 + 6tvy^4 \\
& -6tx^2y^3 + 12tx^3yz - 6tyz^4 - 6u^4xz - 3u^4y^2 \\
& +6u^3v^2z + 12u^3vxy + 2u^3x^3 + 2u^3z^3 - 6u^2v^3y \\
& -9u^2v^2x^2 - 18u^2vyz^2 + 9u^2x^2z^2 + 3u^2y^4 \\
& +6uv^4x + 18uv^2y^2z - 12uvxy^3 - 6ux^4z \\
& +6ux^3y^2 - 6uxz^4 + 6uy^2z^3 + 3v^4z^2 - 12v^3xyz \\
& -2v^3y^3 + 6v^2x^3z + 9v^2x^2y^2 - 3v^2z^4 - 6vx^4y \\
& +12vxyz^3 - 6vy^3z^2 \\
& +2x^3z^3 - 9x^2y^2z^2 + 6xy^4z;
\end{aligned}
$$

Comparison

For simplicity, we denote $d(n) = \mathcal{D}et(C_n)$ and $p(n) = \mathcal{P}(C_n)$.

n	$d(n)$	$p(n)$	n	$d(n)$	$p(n)$
1	1	1	7	246	246
2	2	2	8	810	810
3	4	4	9	2704	2704
4	10	10	10	7492	9252
5	26	26	11	32066	32066
6	68	80	12	86500	112720

Determinant of the Cayley table

Using the theory of **symmetric functions** and p-adic analysis, it is proved that

$$\mathcal{D}et(C_n) = \mathcal{P}(C_n) \quad \Leftrightarrow \quad n \text{ is a prime power.}$$

- Hugh Thomas (J. Algebraic Combin. 2004)
- Colarte, Mezzetti, Miró-Roig and Salat (Proc. Amer. Math. Soc. 2019)

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Theorem 3.3 (Wang-Zhang, 2026)

Let G be a finite abelian group and $|G|$ is a prime power, then

$$\mathcal{D}et(G) = \mathcal{P}(G).$$

This answers a problem of Panyushev (J. Algebraic Combin. 2011).

Transversal

A **transversal** in a Latin square is a collection of cells which do not share any row, column, or symbol.

1	4	6	5	3	2
5	**2**	4	3	1	6
6	3	2	**4**	5	1
2	5	1	6	4	**3**
3	6	**5**	1	2	4
4	1	3	2	**6**	5

Transversal

Enumerating transversals in the cyclic Latin square $\mathcal{M}_{C_n}$ is an important topic in combinatorics, where n is odd.

$$\mathcal{M}_{C_5} = \begin{pmatrix} x_0 & x_1 & \color{red}{x_2} & x_3 & x_4 \\ x_1 & x_2 & x_3 & \color{red}{x_4} & x_0 \\ x_2 & x_3 & x_4 & x_0 & \color{red}{x_1} \\ \color{red}{x_3} & x_4 & x_0 & x_1 & x_2 \\ x_4 & \color{red}{x_0} & x_1 & x_2 & x_3 \end{pmatrix}$$

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There are 15 transversals in $\mathcal{M}_{C_5}$ and 133 transversals in $\mathcal{M}_{C_7}$.

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There are 15 transversals in $\mathcal{M}_{C_5}$ and 133 transversals in $\mathcal{M}_{C_7}$. (Note that, there is no transversal in $\mathcal{M}_{C_n}$ when n is even)

Transversal

Verifying a conjecture of Wanless (London Math. Soc. Lecture Note, 2011), Eberhard, Manners, and Mrazović obtained the following remarkable result using Fourier analytic method (a variant of Hardy-Littlewood circle method).

Theorem. (Eberhard-Manners-Mrazović, J. Eur. Math. Soc. 2019)

Let n be an odd integer, the number of transversals in $\mathcal{M}_{C_n}$ is

$$(e^{-1/2} + o(1))\frac{n!^2}{n^{n-1}}.$$

S. Eberhard, F. Manners and R. Mrazović, *Additive triples of bijections, or the toroidal semiqueens problem*, J. Eur. Math. Soc., (JEMS) 21 (2019), no. 2, 441–463.

Coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\mathsf{per}(\mathcal{M}_{C_n})$

The number of transversals in $\mathcal{M}_{C_n}$ is also coefficient of $x_0 x_1 \cdots x_{n-1}$ in $\mathsf{per}(\mathcal{M}_{C_n})$.

$$\begin{aligned}
\mathsf{per}(\mathcal{M}_{C_5}) &= \mathsf{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 & x_0 \\ x_2 & x_3 & x_4 & x_0 & x_1 \\ x_3 & x_4 & x_0 & x_1 & x_2 \\ x_4 & x_0 & x_1 & x_2 & x_3 \end{pmatrix} \\
&= x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + 5x_3x_4^3x_0 + 5x_3^2x_4x_0^2 + 5x_3^2x_4^2x_1 \\
&\quad + 5x_3^3x_0x_1 + 5x_4x_0^3x_1 + 5x_4^2x_0x_1^2 + 5x_4x_0^2x_1^2 + 5x_3x_4x_1^3 + 5x_3^3x_4x_2 \\
&\quad + 5x_4^2x_0^2x_2 + 5x_3x_0^3x_2 + 5x_4^3x_1x_2 + 5x_3^2x_1^2x_2 + 5x_4x_1^2x_2^2 + 5x_0x_1^3x_2 \\
&\quad + 5x_3x_4^2x_2^2 + 5x_3^2x_0x_2^2 + 5x_0^2x_1x_2^2 + 5x_4x_0x_2^3 + 5x_3x_1x_2^3 \\
&\quad + 15x_0x_1x_2x_3x_4.
\end{aligned}$$

Coefficient of $x_0 x_1 \cdots x_6$ in $\mathsf{per}(\mathcal{M}_{C_7})$

$$\mathsf{per}(\mathcal{M}_{C_7}) = \mathsf{per}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_0 \\ x_2 & x_3 & x_4 & x_5 & x_6 & x_0 & x_1 \\ x_3 & x_4 & x_5 & x_6 & x_0 & x_1 & x_2 \\ x_4 & x_5 & x_6 & x_0 & x_1 & x_2 & x_3 \\ x_5 & x_6 & x_0 & x_1 & x_2 & x_3 & x_4 \\ x_6 & x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}$$

$$= x_0^7 + x_1^7 + x_2^7 + x_3^7 + x_4^7 + x_5^7 + x_6^7 + 7x_0^5 x_1 x_6 + 7x_0^4 x_2 x_6^2 + \cdots$$
$$+ 14x_0^4 x_1 x_2 x_4 + 14x_0^3 x_2^2 x_5^2 + \cdots + 21x_0^3 x_1 x_3 x_5^2 + 21x_0^2 x_1^3 x_5 x_6 + \cdots$$
$$+ 35x_0^3 x_2 x_3 x_4 x_5 + \cdots + 42x_0^2 x_3^2 x_4 x_5 x_6 + \cdots + 49x_0^2 x_2 x_4^2 x_5 x_6 + \cdots$$
$$+ 133x_0 x_1 x_2 x_3 x_4 x_5 x_6.$$

On the coefficients of the group permanent

Mrazović (Thesis, Oxford University 2016) also proposed the problem of studying **coefficient of other monomials**, how is their distribution?

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Eberhard studied that a similar distribution problem for **Latin hypercube** (not the Latin square in our case) and showed that the distribution is almost flat. (completely different from the Latin square case)

S. Eberhard, *More on Additive triples of bijections*, arXiv:1704.02407.

S. Eberhard, F. Manners and R. Mrazović, *An asymptotic for the Hall-Paige conjecture*, Adv. Math. 404 (2022), Paper No. 108423, 73 pp.

On the coefficients of the group permanent

Following and modifying the above approach of Eberhard-Manners-Mrazović, we considered some other monomials.

Theorem 3.4 (Tang-Wu-Zhang, 2026)

When n is odd, the transversal

$$x_0 x_1 \cdots x_{n-1}$$

has the **largest coefficient** *among all the monomials in* $\mathsf{per}(\mathcal{M}_{C_n})$

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In fact, the theorem holds for abelian groups of odd orders.

On the coefficients of the group permanent

Theorem. (Tang-Wu-Zhang, 2025+)

Let n be an odd integer, the coefficient of $x_0^3 x_2 x_3 \cdots x_{n-2}$ in $\mathsf{per}(\mathcal{M}_{C_n})$ is

$$(\frac{1}{3}e^{-1/2} + o(1))\frac{n!^2}{n^{n-1}}.$$

On the coefficients of the group permanent

Theorem. (Tang-Wu-Zhang, 2025+)

Let n be an odd integer, k a fixed integer and $a_i \geq 2$ $(1 \leq i \leq k)$, the coefficient of

$$x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} x_{i_{k+1}} \cdots x_{i_t}$$

in $\mathrm{per}(\mathcal{M}_{C_n})$ is

$$\left(\frac{1}{a_1! \cdots a_k!} e^{-1/2} + o(1)\right) \frac{n!^2}{n^{n-1}}.$$

Immanants

Recall that, for an $n \times n$ matrix $\mathcal{M} = (m_{ij})_{1 \leq i,j \leq n}$, the permanent and the determinant

$$\mathsf{det}(\mathcal{M}) = \sum_{\tau \in S_n} \mathsf{sgn}(\tau) \prod_{i=1}^{n} m_{i,\tau(i)}$$

and

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$$\mathsf{imm}_\lambda(\mathcal{M}) = \sum_{\tau \in S_n} \chi^\lambda(\tau) \prod_{i=1}^{n} m_{i,\tau(i)},$$

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$$\mathsf{imm}_\lambda(\mathcal{M}) = \sum_{\tau \in S_n} \chi^\lambda(\tau) \prod_{i=1}^{n} m_{i,\tau(i)},$$

where χ^λ is an irreducible character of S_n indexed by the partition λ of n.

Note that, $\mathsf{imm}_{(1^n)}(\mathcal{M}) = \mathsf{det}(\mathcal{M})$, $\mathsf{imm}_{(n)}(\mathcal{M}) = \mathsf{per}(\mathcal{M})$.

Immanants

The history of immanants goes back to Schur's work on Hermitian forms and characters, he considered generalized matrix functions associated with group characters and proved fundamental inequalities for positive semidefinite Hermitian matrices A: $\mathsf{imm}_\lambda(A) \geq \chi^\lambda(1)\mathsf{det}(A)$.

The word and systematic study of immanants entered the literature through Littlewood and Richardson's work, their papers placed immanants in the representation theory of symmetric groups and in the emerging language of symmetric functions.

Immanants

Classical works on immanants developed along several directions:

(1) Elliott H. Lieb's Permanental dominance conjecture:

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(3) Positivity phenomena on totally positive or totally nonnegative matrices: J. R. Stembridge, Immanants of totally positive matrices are nonnegative, Bull. London Math. Soc. 23, no. 5 (1991), 422–428.

Immanants - computational complexity

For $\lambda \vdash n$ and the irreducible character χ^λ of S_n,

$$\mathsf{imm}_\lambda(A) = \sum_{\pi \in S_n} \chi^\lambda(\pi) \prod_{i=1}^{n} A_{i,\pi(i)}.$$

Easy: determinant

$$\lambda = (1^n), \quad \chi^\lambda = \mathrm{sgn}, \quad \mathsf{imm}_\lambda = \det.$$

Gaussian elimination gives polynomial-time computation; algebraically, the determinant family lies in VP.

Hard: permanent

$$\lambda = (n), \quad \chi^\lambda = 1, \quad \mathsf{imm}_\lambda = \mathrm{per}.$$

Valiant proved the permanent of 0–1 matrices is #P-complete; it is also the canonical VNP-complete family.

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determinant **immanants** **permanent**
a bridge

L. G. Valiant, STOC 1979.

Representative results: a dichotomy

Bürgisser (SIAM J. Comput., 2000), Bürgisser-Ikenmeyer-Panova (J. Amer. Math. Soc. 2019)

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- For hook diagrams and rectangular diagrams of polynomially growing width,

$$\mathrm{Imm}_\lambda \text{ is \#P-complete and VNP-complete.}$$

- Hence many immanants between determinant and permanent are already permanent-hard.

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Curticapean (STOC 2021): a full complexity **dichotomy**

Let $b(\lambda) = n - \ell(\lambda)$, the number of boxes to the right of the first column. For a family Λ:

$$\begin{aligned} b(\Lambda) < \infty &\implies \text{polynomial-time evaluation,} \\ b(\Lambda) \text{ unbounded} &\implies \text{intractability under standard parameterized assumptions,} \\ b(\lambda) \text{ polynomially growing} &\implies \text{\#P- and VNP-hardness.} \end{aligned}$$

Immanants of the Cayley table

Now, we consider the immanants

$$\mathsf{imm}_\lambda(\mathcal{M}) = \sum_{\tau \in S_n} \chi^\lambda(\tau) \prod_{i=1}^{n} m_{i,\tau(i)},$$

where χ^λ is an irreducible character of S_n indexed by the partition λ of n.

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We consider the partition $\lambda = (n-1, 1)$. Let $\mathsf{Fix}(\sigma)$ be the number of fixed points of $\sigma \in \mathsf{S}_n$. For example, if $\sigma = (134)(2)(57)(6) \in \mathsf{S}_7$, then $\mathsf{Fix}(\sigma) = 2$.

Then we have

$$\chi^{(n-1,1)}(\sigma) = \mathsf{Fix}(\sigma) - 1.$$

Immanants of the Cayley table

Let $\mathcal{I}_\lambda(n)$ be the number of formally different monomials occurring in $\mathsf{imm}_\lambda(\mathcal{M}_{C_n})$.

Recall that $p(n) = \mathcal{P}(C_n)$, $d(n) = \mathcal{D}et(C_n)$, and $p(n) = d(n)$ iff n is a prime power.

n	$\mathsf{d}(n)$	$\mathsf{p}(n)$	$\mathcal{I}_{(n-1,1)}(n)$	n	$\mathsf{d}(n)$	$\mathsf{p}(n)$	$\mathcal{I}_{(n-1,1)}(n)$
2	2	2	2	7	246	246	0
3	4	4	0	8	810	810	502
4	10	10	6	9	2704	2704	0
5	26	26	0	10	7492	9252	9252
6	68	80	80	11	32066	32066	0

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Several numerical results and partial results were obtained in

X. Wang, H. Zhang and S. Zhang, *On Immanants of the Cayley Table of Finite Abelian Groups*, Bull. Braz. Math. Soc. (2026).

Immanants of the Cayley table

For abelian group G, let $\mathcal{I}_\lambda(G)$ be the number of formally different monomials occurring in $\mathsf{imm}_\lambda(\mathcal{M}_G)$.

Theorem 3.5 (Wang-Zhang, 2026)

Let G be an ablian group with $|G| \geq 3$ odd, then

$$\mathcal{I}_{(n-1,1)}(G) = 0.$$

Theorem 3.6 (Wang-Zhang, 2026)

Let G be an abelian group with $|G| \geq 3$ odd, then

$$\mathcal{I}_{(2,1^{n-2})}(G) = 0.$$

Note that $(n-1,1)$ and $(2,1^{n-2})$ are conjugate.

Immanants of the Cayley table

Theorem 3.7 (Wang-Zhang, 2026)

Let G be an abelian group with $|G| \equiv 2 \pmod 4$, then

$$\mathcal{I}_{(n-1,1)}(G) = \mathcal{P}(G).$$

Immanants of the Cayley table

Theorem 3.7 (Wang-Zhang, 2026)

Let G be an abelian group with $|G| \equiv 2 \pmod 4$, then

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Theorem 3.8 (Wang-Zhang, 2026)

Let G be an ablian group with $|G| \equiv 2 \pmod 4$, then

$$\mathcal{I}_{(2,1^{n-2})}(G) = \mathcal{D}et(G).$$

Immanants of the Cayley table

Theorem 3.9 (Wang-Zhang, 2026)

Let G be an abelian group with $|G| \geq 7$ odd, then

$$\mathsf{imm}_{(4,1^{n-4})}(\mathcal{M}_G) = \mathsf{imm}_{(2,2,2,1^{n-6})}(\mathcal{M}_G).$$

Immanants of the Cayley table

Theorem 3.9 (Wang-Zhang, 2026)

Let G be an abelian group with $|G| \geq 7$ odd, then

$$\mathsf{imm}_{(4,1^{n-4})}(\mathcal{M}_G) = \mathsf{imm}_{(2,2,2,1^{n-6})}(\mathcal{M}_G).$$

They are equal as polynomials.

Thanks for your attention!